Problem 13

If you invest a dollar at "6% interest compounded monthly," it amounts to $(1.005)^n$ dollars after n months. If you invest \$10 at the beginning of each month for 10 years (120 months), how much will you have at the end of the 10 years?

Solution

Suppose you invest x dollars at the beginning of month 1: $P(0) = x$. By the end of month 1, the principal will be $P(0)$ plus the earned interest $iP(0)$ plus the newly invested x dollars.

$$P(1) = [P(0) + iP(0)] + x = (2 + i)x$$

By the end of month 2, the principal will be $P(1)$ plus the earned interest $iP(1)$ plus the newly invested x dollars.

$$P(2) = [P(1) + iP(1)] + x = (3 + 3i + i^2)x$$

There doesn't seem to be an obvious pattern to determine $P(n)$, so formulate a boundary value problem for it.

$$P_{n+1} = (1 + i)P_n + x, \quad P_1 = (2 + i)x$$

This is a first-order linear difference equation, which can be solved with a summing factor.

$$S = \frac{1}{\prod_{j=1}^{n}(1+i)} = \frac{1}{(1+i)^n} = (1+i)^{-n}$$

Multiply both sides of the difference equation by S.

$$(1+i)^{-n}P_{n+1} = (1+i)^{1-n}P_n + x(1+i)^{-n}$$

Bring the term with P_n to the left side.

$$(1+i)^{-n}P_{n+1} - (1+i)^{-(n-1)}P_n = x(1+i)^{-n}$$

Write the left side as a discrete derivative. (Note: $DP_n = P_{n+1} - P_n$.)

$$D\left[(1+i)^{-(n-1)}P_n\right] = x(1+i)^{-n}$$

Sum both sides from 1 to $n - 1$.

$$(1+i)^{-(n-1)}P_n - (1+i)^{-(1-1)}P_1 = \sum_{j=1}^{n-1} x(1+i)^{-j}$$

$$(1+i)^{-(n-1)}P_n - P_1 = x\sum_{j=1}^{n-1}\left(\frac{1}{1+i}\right)^j$$

$$(1+i)^{-(n-1)}P_n = P_1 + x\left[-1 + \sum_{j=0}^{n-1}\left(\frac{1}{1+i}\right)^j\right]$$

Evaluate the sum and plug in $P_1 = (2+i)x$.

$$(1+i)^{-(n-1)}P_n = (2+i)x + x\left[-1 + \frac{1-\left(\frac{1}{1+i}\right)^n}{1-\left(\frac{1}{1+i}\right)}\right]$$

Solve for P_n and simplify the result.

$$P_n = \frac{x}{i}\left[(1+i)^{n+1} - 1\right]$$

The fact that a dollar invested at "6% interest compounded monthly" amounts to $(1.005)^n$ dollars after n months means that 6% is an annual interest rate.

$$i = \frac{0.06}{12} = 0.005$$

Therefore, the principal (in dollars) after n months that is compounded monthly at 6% interest and that has \$10 added monthly is

$$P_n = 2010 \times 1.005^n - 2000.$$

After 10 years, or 120 months, it is

$$P_{120} = 2010 \times 1.005^{120} - 2000 \approx \$1656.99.$$

Assuming \$10 is not put in the account at the end of the 120th month, the principal will just be about

$$\$1646.99.$$